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LETTER TO THE EDITOR

An exactly solvable periodic Schrödinger operator

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Abstract. We explicitly determine the band spectrum of a periodically strongly singular Schrödinger operator in $L^2(\mathbb{R})$ associated with the differential expression $-\mathrm{d}^2/\mathrm{d}x^2+(s^2-\frac{1}{4})/\cos^2 x+(s'^2-\frac{1}{4})/\sin^2 x$, $x\in\mathbb{R}\backslash\frac{1}{2}\pi\mathbb{Z}$, $0<s, s'<1$. The corresponding density of states is calculated analytically.

In this letter we are concerned with an explicit determination of the spectrum of a periodically strongly singular Schrödinger operator associated with the differential expression

$$\tau=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{s^2-\frac{1}{4}}{\cos^2 x}+\frac{s'^2-\frac{1}{4}}{\sin^2 x},\qquad x\in\mathbb{R}\backslash\tfrac{1}{2}\pi\mathbb{Z},\ s, s'>0. \tag{1}$$

In order to relate τ with a self-adjoint operator in $L^2(\mathbb{R})$ we introduce the minimal operator $\dot{T}$ in $L^2(\mathbb{R})$,

$$\begin{aligned}&\mathscr{D}(\dot{T})=\{f\in L^2(\mathbb{R})\,|\,f, f'\in AC_{\mathrm{loc}}(\mathbb{R});\ \operatorname{supp} f\subset\mathbb{R}\backslash\tfrac{1}{2}\pi\mathbb{Z}\text{ compact};\ \tau f\in L^2(\mathbb{R})\},\\ &(\dot{T}f)(x)=(\tau f)(x),\qquad x\in\mathbb{R}\backslash\tfrac{1}{2}\pi\mathbb{Z}\end{aligned} \tag{2}$$

where $AC_{\mathrm{loc}}(Y)$ denotes the set of locally absolutely continuous functions on Y. By inspection, the closure T of $\dot{T}$ is symmetric and has infinite deficiency indices if $0<s<1$ or $0<s'<1$ and we are particularly interested in self-adjoint extensions of T. If $s\geqslant 1$ and $s'\geqslant 1$ then T is self-adjoint (cf Gesztesy and Kirsch 1984, from now on referred to as GK). In the following it is also necessary to consider restrictions $\dot{T}_j$, T_j of $\dot{T}$, T to a periodicity interval $I_j\equiv(j\pi-\pi/2, j\pi)\cup(j\pi, j\pi+\pi/2)$. Due to the singularity structure of τ near points in $\frac{1}{2}\pi\mathbb{Z}$ we introduce the boundary values

$$\begin{aligned}
f_{(j\pi+\pi/2)_\pm}&=\mp\lim_{x\to(j\pi+\pi/2)_\pm}\{f'(x)|x-(j\pi+\pi/2)|^{s+1/2}\mp(s+\tfrac{1}{2})f(x)|x-(j\pi+\pi/2)|^{s-1/2}\}\\
f'_{(j\pi+\pi/2)_\pm}&=\lim_{x\to(j\pi+\pi/2)_\pm}\{f'(x)|x-(j\pi+\pi/2)|^{1/2-s}\mp(\tfrac{1}{2}-s)f(x)|x-(j\pi+\pi/2)|^{-1/2-s}\}\\
f_{(j\pi-\pi/2)_\pm}&=\mp\lim_{x\to(j\pi-\pi/2)_\pm}\{f'(x)|x-(j\pi-\pi/2)|^{s+1/2}\mp(s+\tfrac{1}{2})f(x)|x-(j\pi-\pi/2)|^{s-1/2}\}\\
f'_{(j\pi-\pi/2)_\pm}&=\lim_{x\to(j\pi-\pi/2)_\pm}\{f'(x)|x-(j\pi-\pi/2)|^{1/2-s}\mp(\tfrac{1}{2}-s)f(x)|x-(j\pi-\pi/2)|^{-1/2-s}\}\\
f_{(j\pi)_\pm}&=\mp\lim_{x\to(j\pi)_\pm}\{f'(x)|x-j\pi|^{s'+1/2}\mp(s'+\tfrac{1}{2})f(x)|x-j\pi|^{s'-1/2}\}\\
f'_{(j\pi)_\pm}&=\lim_{x\to(j\pi)_\pm}\{f'(x)|x-j\pi|^{1/2-s'}\mp(\tfrac{1}{2}-s')f(x)|x-j\pi|^{-1/2-s'}\}
\end{aligned} \tag{3}$$

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where f is locally in $\mathscr{D}(T_j^*)$ near $\frac{1}{2}\pi\mathbb{Z}$ or $f\in\mathscr{D}(T^*)$ (we recall $\mathscr{D}(T^*)=\{f\in L^2(\mathbb{R})|f,f'\in AC_{\text{loc}}(\mathbb{R}\backslash\frac{1}{2}\pi\mathbb{Z});\ \tau f\in L^2(\mathbb{R})\}$ and similarly for $\mathscr{D}(T_j^*)$ in $L^2(I_j)$). Obviously $f_{(j\pi)_\pm}=f((j\pi)_\pm)$, $f'_{(j\pi)_\pm}=f'((j\pi)_\pm)$ if $s'=\frac{1}{2}$ and analogously for $s=\frac{1}{2}$. We start with the following theorem.

Theorem 1. Assume $0<s,\ s'<1$. Then the operators

$$\begin{aligned}\mathscr{D}(T_j^\theta)=\{f\in L^2(I_j)\,|\,f,f'\in AC_{\text{loc}}(I_j);\ &f_{(j\pi-\pi/2)_+}=\mathrm{e}^{\mathrm{i}\theta}f_{(j\pi+\pi/2)_-},\\ &f'_{(j\pi-\pi/2)_+}=\mathrm{e}^{\mathrm{i}\theta}f'_{(j\pi+\pi/2)_-},\ f_{(j\pi)_-}=f_{(j\pi)_+},\\ &f'_{(j\pi)_-}=f'_{(j\pi)_+};\ \tau f\in L^2(I_j)\},\end{aligned}\tag{4}$$

$$(T_j^\theta f)(x)=(\tau f)(x),\ x\in I_j,\ 0\leqslant\theta<2\pi$$

are self-adjoint extensions of T_j in $L^2(I_j)$. The spectra of these operators are purely discrete and given by

$$\sigma(T_j^\theta)=\begin{cases}\{\pi^{-1}[\cos^{-1}(\sin(\pi s)\sin(\pi s')\cos\theta-\cos(\pi s)\cos(\pi s'))]+2n\}^2, & \theta\in(0,\pi),\\ [1\mp(s+s')+2n]^2, & \theta=0,\\ [1\mp(s-s')+2n]^2, & \theta=\pi,\qquad n=0,1,2,\ldots,\end{cases}\tag{5}$$

$$\sigma(T_j^{2\pi-\theta})\sigma(T_j^\theta),\ \theta\in[0,\pi].\tag{6}$$

Proof. Since the first part results from the general treatment in GK we only need to prove (5) and (6). By inspection the general solution ψ of the equation

$$(\tau\psi)(k,x)=k^2\psi(k,x),\ k^2\in\mathbb{R},\ \operatorname{Im}k\geqslant0,\ x\in I_j\tag{7}$$

is

$$\psi(k,x)=\begin{cases}c_1^+\psi_1(k,x)+c_2^+\psi_2(k,x), & x\in(j\pi,j\pi+\pi/2)\\ c_1^-\psi_1(k,x)+c_2^-\psi_2(k,x), & x\in(j\pi-\pi/2,j\pi),\end{cases}\tag{8}$$

$$\begin{aligned}\psi_1(k,x)&=(\sin^2x)^{1/4-s'/2}(\cos^2x)^{1/4-s/2}{}_2F_1(a;b;c;\sin^2x),\\ \psi_2(k,x)&=(\sin^2x)^{1/4+s'/2}(\cos^2x)^{1/4-s/2}{}_2F_1(a-c+1;b-c+1;2-c;\sin^2x)\end{aligned}\tag{9}$$

where ${}_2F_1(\alpha;\beta;\gamma;z)$ denotes the hypergeometric function (Abramowitz and Stegun 1972) and

$$a=\tfrac{1}{2}(k-s-s'+1),\qquad b=\tfrac{1}{2}(-k-s-s'+1),\qquad c=1-s'.\tag{10}$$

Analytic continuation $\sin^2x\to\cos^2x$ reads (Abramowitz and Stegun 1972)

$$\begin{aligned}\psi_1(k,x)=E&(\sin^2x)^{1/4-s'/2}(\cos^2x)^{1/4-s/2}{}_2F_1(a;b;a+b+1-c;\cos^2x)\\ &+F(\sin^2x)^{1/4-s'/2}(\cos^2x)^{1/4+s/2}{}_2F_1(c-b;c-a;c-a-b+1;\cos^2x),\\ \psi_2(k,x)=G&(\sin^2x)^{1/4+s'/2}(\cos^2x)^{1/4-s/2}{}_2F_1(a;b;a+b+1-c;\cos^2x)\\ &+H(\sin^2x)^{1/4+s'/2}(\cos^2x)^{1/4+s/2}{}_2F_1(c-b;c-a;c-a-b+1;\cos^2x)\end{aligned}\tag{11}$$

where

$$E=\frac{\Gamma(s)\Gamma(1-s')}{\Gamma[(-k+s-s'+1)/2]\Gamma[(k+s-s'+1)/2]},$$

$$
\begin{aligned}
F &= \frac{\Gamma(-s)\Gamma(1-s')}{\Gamma[(k-s-s'+1)/2]\Gamma[(-k-s-s'+1)/2]},\\
G &= \frac{\Gamma(s)\Gamma(1+s')}{\Gamma[(-k+s+s'+1)/2]\Gamma[(k+s+s'+1)/2]},\\
H &= \frac{\Gamma(-s)\Gamma(1+s')}{\Gamma[(k-s+s'+1)/2]\Gamma[(-k-s+s'+1)/2]}.
\end{aligned}
\tag{12}
$$

A lengthy although straightforward computation then shows

$$
\begin{aligned}
&\psi_{(j\pi-\pi/2)_+} = 2s(c_1^- E + c_2^- G), \qquad \psi'_{(j\pi-\pi/2)_+} = 2s(c_1^- F + c_2^- H),\\
&\psi_{(j\pi+\pi/2)_-} = 2s(c_1^+ E + c_2^+ G), \qquad \psi'_{(j\pi+\pi/2)_-} = -2s(c_1^+ F + c_2^+ H),\\
&\psi_{j\pi_+} = 2s'c_1^+, \qquad \psi'_{j\pi_+} = 2s'c_2^+, \qquad \psi_{j\pi_-} = 2s'c_1^-, \qquad \psi'_{j\pi_-} = -2s'c_2^-.
\end{aligned}
\tag{13}
$$

Imposing the boundary conditions in (4) finally yields

$$
\begin{aligned}
&c_1^+ = c_1^-, \qquad c_2^+ = -c_2^-,\\
&c_1^- E + c_2^- G = e^{i\theta}(c_1^+ E + c_2^+ G), \qquad c_1^- F + c_2^- H = -e^{i\theta}(c_1^+ F + c_2^+ H).
\end{aligned}
\tag{14}
$$

Employing (12) this is equivalent to (5). Formula (6) follows since T_j^θ and $T_j^{2\pi-\theta}$, $\theta \in [0, \pi]$ are anti-unitarily equivalent (cf Reed and Simon 1978).

If $s > 1$ (or $s' > 1$) then T_j automatically obeys a Dirichlet boundary condition near $j\pi \pm \pi/2$ (or $j\pi$).

The machinery of direct integral decomposition for periodic Schrödinger operators then yields the theorem below.

Theorem 2. Let $0 < s,\ s' < 1$. Then the operator $H_{s,s'}$ in $L^2(\mathbb{R})$,

$$
\begin{aligned}
\mathscr{D}(H_{s,s'}) = \{&f \in L^2(\mathbb{R}) \,|\, f, f' \in AC_{\mathrm{loc}}(\mathbb{R}\backslash\tfrac{1}{2}\pi\mathbb{Z});\ f_{(y_j)_-} = f_{(y_j)_+},\\
&f'_{(y_j)_-} = f'_{(y_j)_+},\ y_j \in \tfrac{1}{2}\pi\mathbb{Z};\ \tau f \in L^2(\mathbb{R})\}
\end{aligned}
\tag{15}
$$

$$(H_{s,s'}f)(x) = (\tau f)(x),\ x \in \mathbb{R}\backslash\tfrac{1}{2}\pi\mathbb{Z}$$

is self-adjoint and unitarily equivalent to the direct integral $\int_{[0,2\pi)}^{\oplus} T^\theta\, d\theta/2\pi$ where T^θ denotes the family of self-adjoint operators in $L^2((-\pi/2, \pi/2))$,

$$
\begin{aligned}
\mathscr{D}(T^\theta) = \{&g(\theta) \in L^2((-\pi/2, \pi/2)) \,|\, g(\theta), g'(\theta) \in AC_{\mathrm{loc}}((-\pi/2, \pi/2)\backslash\{0\});\\
&g(\theta)_{(-\pi/2)_+} = e^{i\theta} g(\theta)_{(\pi/2)_-},\ g'(\theta)_{(-\pi/2)_+} = e^{i\theta} g'(\theta)_{(\pi/2)_-},\\
&g(\theta)_{(0)_+} = g(\theta)_{(0)_-};\ \tau g(\theta) \in L^2((-\pi/2, \pi/2))\},\\
&(T^\theta g(\theta))(x) = (\tau g(\theta))(x),\ x \in (-\pi/2, \pi/2)\backslash\{0\},\ 0 \leqslant \theta < 2\pi
\end{aligned}
\tag{16}
$$

(prime denotes differentiation with respect to x). The operator $H_{s,s'}$ has no eigenvalues and

$$
\begin{aligned}
\sigma(H_{s,s'}) = &\bigcup_{n=0}^{\infty} [(2n+1-s-s')^2, (2n+1+s-s')^2]\\
&\cup \bigcup_{n=0}^{\infty} [(2n+1-s+s')^2, (2n+1+s+s')^2]
\end{aligned}
\tag{17}
$$

if $0<s\leq s'<1$ and $s+s'<1$,

$$\sigma(H_{s,s'})=[(1-s-s')^2,(1+s-s')^2]\cup\bigcup_{n=0}^{\infty}[(2n+1-s+s')^2,(2n+3-s-s')^2]$$
$$\cup\bigcup_{n=0}^{\infty}[(2n+1+s+s')^2,(2n+3+s-s')^2] \qquad (18)$$

if $0<s\leq s'<1$ and $s+s'>1$,

$$\sigma(H_{s,s'})=[0,(1-s-s')^2]\cup\bigcup_{n=0}^{\infty}[(2n+1-s+s')^2,(2n+3+s-s')^2] \qquad (19)$$

if $0<s\leq s'<1$ and $s+s'=1$. If $0<s'\leq s<1$ one simply exchanges s and s' in (17)-(19).

Since one can follow the proof of theorem 4.3 in GK step by step we omit the details.

Remark 3. (*a*) The cases $s=\frac{1}{2}$ or $s'=\frac{1}{2}$ have been discussed by Scarf (1958) (assuming the validity of Bloch's theorem). A rigorous proof of this case appeared in GK. The special case $0<s=s'<1$ also subordinates to this treatment.

(*b*) For $s+s'\neq 1$ all gaps occur since $E_n(0)$, $E_n(\pi)$, $n\in\mathbb{N}_0(E_n(\theta)$ the eigenvalues of T_j^θ listed in (5)) are simple. For $s+s'=1$, $s'\neq\frac{1}{2}$ every second gap closes since $E_0(0)$, $E_n(\pi)$, $n\in\mathbb{N}_0$ are simple but $E_n(0)$, $n\in\mathbb{N}$ are twice degenerate. For $s=s'=\frac{1}{2}$ all gaps close since $E_0(0)$ is simple but $E_n(0)$, $n\in\mathbb{N}$ and $E_n(\pi)$, $n\in\mathbb{N}_0$ are twice degenerate. In fact $\sigma(H_{1/2,1/2})=[0,\infty)$ since $H_{1/2,1/2}=-\mathrm{d}^2/\mathrm{d}x^2$ on $H^{2,2}(\mathbb{R})$.

(*c*) If $s>1$ (or $s'>1$) then $\sigma(H_{s,s'})$ is pure point with each eigenvalue of infinite multiplicity since there are Dirichlet boundary conditions at points $(2j+1)\pi/2$, $j\in\mathbb{Z}$ (or at $\pi\mathbb{Z}$).

(*d*) The reason why the above model can be treated analytically is connected with its complete integrability as a classical as well as quantum system as discussed e.g. in Calogero (1975) and Olshanetsky and Perelomov (1983). Next we note that the density of states $\mathrm{d}\rho_{s,s'}/\mathrm{d}E$ of $H_{s,s'}$ at a point E with $E_n(\theta)=E$, $n\in\mathbb{N}_0$ is given by (cf (5))

$$\begin{aligned}\mathrm{d}\rho_{s,s'}/\mathrm{d}E &= \pi^{-2}\,\mathrm{d}\theta/\mathrm{d}E\\ &=\{\sin^2(\pi s)\sin^2(\pi s')-[\cos(\pi E^{1/2})+\cos(\pi s)\cos(\pi s')]^2\}^{-1/2}\\ &\quad\times(2\pi)^{-1}E^{-1/2}\sin(\pi E^{1/2}), \qquad E\in\mathring{\sigma}(H_{s,s'})\end{aligned} \qquad (20)$$

($\mathring{A}$ the interior of A). Obviously

$$\mathrm{d}\rho_{s,s'}/\mathrm{d}E \xrightarrow[s,s'\to 1/2]{} (2\pi)^{-1}E^{-1/2},\ E\in\bigcap_{s,s'}\mathring{\sigma}(H_{s,s'})\cap(0,\infty) \qquad (21)$$

where the RHS of (21) represents the (unperturbed) density of states of $H_{1,2,1/2}=-\mathrm{d}^2/\mathrm{d}x^2$. We finally observe that $\mathrm{d}\rho_{s,s'}/\mathrm{d}E$ exhibits the usual $|E-a_n|^{-1/2}$ singularities near the band edges a_n, $n\in\mathbb{N}_0$. As an example consider e.g. the case $0<s<\frac{1}{2}<s'<1$, $s+s'=1$ and $E\to 0_+$ where

$$\mathrm{d}\rho_{s,s'}/\mathrm{d}E \underset{E\to 0_+}{=} [1+\cos(\pi s)\cos(\pi s')]^{-1/2}(2\pi)^{-1}E^{-1/2}+\mathrm{O}(E^{1/2})$$
$$0<s<\tfrac{1}{2}<s'<1,\ s+s'=1. \qquad (22)$$

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